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MODELS OF ARITHMETIC AND RECURSIVE FUNCTIONS

BY

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ABSTRACT

We investigate homomorphic images of the semiring of recursive functions as models of the Π_2 fragment of Arithmetic, and some relations between this fragment, its models and recursion theory.

From the point of view of model theory, Arithmetic is a very complicated theory for which non-standard models cannot be described in any "constructive" way. On the other hand, any systematic research in number theory concerns only a limited fragment of the whole theory and most of the theorems in a standard textbook lie very low in the Arithmetical hierarchy. This leads us to the attempt to specify a fragment of Arithmetic which is strong enough to include most of "number theory" and for which model theory may add some insight.

In this paper we study the theory T — the Π_2 fragment of Arithmetic — and its models. It turns out that the theory and its models are closely related to recursion theory. The notions of recursion theory extend naturally to T and in view of Matijasevic's result recursive functions are absolute in these models. Indeed, being a model of T amounts to being closed under recursive functions. From the model theoretic point of view, T is inductive (the union of a chain of models is again a model) and the intersection of a family of models is also a model.

In Section 2, we turn to the main subject—homomorphic images of the semiring of recursive functions. These models were suggested by Feferman, Scott and Tennenbaum [1] and investigated by M. Lerman [5, 6]. It was showm there that such models fail to satisfy an instance of the axiom of induction. We show that if not for trivial reasons (like having zero divisors), such homomorphic images are what we call recursive ultrapowers and they are models of T. (Recursive ultrapowers are constructed similarly to general ultrapowers using

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only recursive sets and recursive functions). On the other hand, we describe a formula that defines the set of natural numbers in any recursive ultrapower.

The recursive ultrapowers are the basic models from which all models of T are composed (in a sense that will be made precise in Section 3). But at the same time every countable model of T (and in particular every countable model of full Arithmetic) can be embedded in such a homomorphic image of the recursive functions. Moreover, using an idea of H. J. Keisler [4], we characterize the class of countable models of T as the class of limit recursive ultrapowers.

Section 4 discusses briefly some examples of recursive ultrapowers including those generated by a comaximal set. We disprove a conjecture by A. Narode (that such models have theories which are arithmetical sets) and improve a result by M. Lerman which gives a condition under which elementary equivalence of models implies isomorphism.

In conclusion, Section 5 outlines a similar construction for full Arithmetic. The class of countable models of this theory is characterized as the class of limit arithmetical ultrapowers.

1. The Π_2 fragment of arithmetic

We work in the language that has the symbols $0, 1, +, \cdot$ and $<$. A formula in this language is *bounded* if all its quantifiers are of the form $\forall x < t$ or $\exists x < t$ (where $\forall x < t\phi$ is $\forall x(x < t \rightarrow \phi$ and $\exists x < t\phi$ is $\exists x(x < t \wedge \phi)$). Σ_1 formulas are obtained from bounded ones by prefixing a string of existential quantifiers and Π_1 fomulas are obtained similarly prefixing a string of universal quantifiers. More generally, if ϕ is in Σ_n and ψ is in Π_n then $\forall \bar{x}\phi$ is in Π_{n+1} and $\exists \bar{x}\psi$ is in Σ_{n+1} (where $\forall \bar{x}$ is an abbreviation of $\forall x_1 \cdots \forall x_n$).

Throughout this paper we denote by T the Π_2 fragment of Arithmetic — all the Π_2 statements that hold in the set of natural numbers N.

1.1 Every model M of T includes N and is lineary ordered by $<$. This relation satisfies

$$x < y \leftrightarrow \exists z(x + z + 1 = y)$$

and N is an initial segment of M.

1.2 A formula which is equivalent in T to a Σ_1 formula is called an *r.e. formula*. An r.e. formula whose negation is also r.e. is a *recursive formula*. If two such formulas describe the same predicate in N they do so in every model of T. (Note that a formula which is not Σ_1 may describe an r.e predicate on N without being r.e by our definition. But it is still the case that every r.e predicate on N has also Σ_1 formulas which describe it.) If a recursive formula describes (is the graph of) a function on N, it does so in every model of T. We call this function a *recursive function*. A recursive function, $f(\bar{x})$, may be substituted in a predicate $\psi(z)$ to obtain the predicate $\psi(f(\bar{x}))$. By this we mean the predicate $\exists y(\phi(\bar{x},y)\wedge\psi(y))$ where ϕ is any recursive formula that describes $f(\bar{x})$. As f is a total function $\psi(f(\bar{x}))$ is equivalent in T to $\forall y(\phi(\bar{x},y)\rightarrow\psi(y))$. From this, it follows that if $\psi(z)$ is recursive so is also $\psi(f(\bar{x}))$. This definition in addition takes care of composition of recursive functions.

1.3 It is not hard to check that recursive formulas are closed under the prefixing of bounded quantifiers (the analogue for r.e formulas is false; the required equivalence to a Σ_1 formula does not always hold in T). Thus, we have in T the axiom of induction for recursive formulas: If $\phi(x,\bar{y})$ is recursive then

$$T\vdash\forall\bar{x}(\exists y\phi(\bar{x},y)\rightarrow\exists y[\phi(\bar{x},y)\wedge\forall z<y\ \neg\phi(\bar{x},z)])$$

(we can show that this may be false if ϕ is only assumed to be Σ_1 and we do not know what the case is for Π_1 formulas).

1.4 Kleene's enumeration theorem has a Π_2 form. We shall need both forms — for r.e predicates and for partial recursive functions:

a) For every n there is an $n+1$ place Σ_1 predicate $F^n(y,\bar{x})$ such that for every n-place Σ_1 predicate $\phi(\bar{x})$ we can find some $i\in N$ for which:

$$T\vdash\forall\bar{x}(F^n(i,\bar{x})\leftrightarrow\phi(\bar{x},z)).$$

b) There is an $n+2$ place Σ_1 predicate $V^n(y,\bar{x},z)$ which describes a partial function $z=f(y,\bar{x})$ such that if $\phi(\bar{x},z)$ is a Σ_1 predicate which describes a partial function then for some $i\in N$:

$$T\vdash\forall\bar{x}\ \forall z(V^n(i,\bar{x},z)\leftrightarrow\phi(\bar{x},z)).$$

1.5 Matijasevic's theorem [7] enables us to relate the notions of recursion theory to simple model theoretic properties. It claims that in N, every Σ_1 formula is equivalent to an existential formula. As the equivalent has a Π_2 normal form it holds also in T.

1.6 The Σ_1 formulas are preserved under extensions in the class of models of T. Recursive formulas are absolute in this class as are the recursive functions in particular. Indeed, it is easy to see that the absoluteness completely characterizes the recursive predicates (this was first observed by H. Gaifman in [2]).

1.7 We show that T is an inductive theory—the union of a chain of models of T is again a model of T.

LEMMA. *Let $\{M_i \mid i \in I\}$. Then for any Σ_1 formula $\phi(\bar{x})$ and parameters $\bar{a} \in M$, $M \models \phi(\bar{a})$ iff there is some i such that for all $i < j$ $M_j \models \phi(\bar{a})$.*

PROOF. The proof is by induction on the number of quantifiers. The statement clearly holds for quantifier-free formulas. Assume that $\psi = \exists x \phi(\bar{a}, x)$ and $M \models \psi$ (the quantifier may be bounded or not). Then $M \models \phi(\bar{a}, b)$ for some $b \in M$ and the claim follows from the induction assumption. If, on the other hand, $M_j \models \exists x \phi(\bar{a}, x)$ from some i on then we can fix one such j_0 and $M_{j_0} \models \phi(\bar{a}, b)$ for some $b \in M_{j_0}$. By 1.6, $M_j \models \phi(\bar{a}, b)$ for every $j_0 < j$ and again by the induction assumption $M \models \phi(\bar{a}, b)$.

If the first quantifier is universal, ψ must be a bounded formula. But then, $\neg\psi$ is also Σ_1 with the same number of quantifiers and the first one is existential. Thus, the proof above applies to $\neg\psi$, and the claim for ψ follows, by 1.6.

q.e.d.

1.7.1. COROLLARIES

a. T is an inductive theory.

b. If we denote by S the set of AE statements of Arithmetic (i.e.—existential formulas preceeded by block of universal quantifiers), then $S \equiv T$.

Part *a.* follows immediately from the lemma and the implication $(a) \Rightarrow (b)$ is a theorem by Chang, Los and Suszko [9, p. 77].

1.8. THEOREM. *Let M be a model of T and $A \subset M$. Then A is (the domain of) a model of T iff A is closed under recursive functions (as defined in M).*

PROOF. If A is itself a model of T, then it is closed under recursive functions by 1.6.

Assume that A is closed under recursive functions and let ϕ be in T. By 1.7.1, we can assume that $\phi = \forall \bar{x} \exists y_1 \cdots y_k \psi(\bar{x}, y_1, \cdots, y_k)$ where ψ is quantifier free. It is easy to construct recursive functions $f_1(\bar{x}), \cdots, f_k(\bar{x})$ such that $N \models \forall \bar{x} \phi(\bar{x}, f_1(\bar{x}), \cdots, f_k(\bar{x}))$. This statement (where the functions are represented by recursive formulas) transfers to M. As A is closed under recursive functions, we get for every $\bar{a} \in A$ elements $b_1, \cdots, b_k \in A$ such that $M \models \psi(\bar{a}, b_1 \cdots b_k)$, and as ψ is quantifier free the same is true in the model whose domain is A. Hence, it is also a model of ϕ.

q.e.d.

Recursive functions are closed under compositions thus:

1.8.1. Corollary. *If $M \models T$ and $B \subset M$ then there is a minimal submodel of M which includes B and is a model of T. This is the closure of B under recursive functions.*

As a closure under functions is preserved by intersections, we have also by 1.8:

1.8.2. Corollary. *T is closed under intersections: If $\{M_i \mid i \in I\}$ are all models of T which are submodels of a model M of T, then their intersection is also a model of T.*

2. Recursive ultrapowers

2.1 A *recursive ultrafilter* is a maximal collection of recursive sets of N with the finite intersection property. If A is recursive then its complement $\bar{A}$ is also recursive. Therefore, the maximality condition is equivalent to the statement that for every recursive set either it, or its complement, is in the filter.

Let R be the semiring of recursive functions and let F be a recursive ultrafilter. We define a relation on R

$$f \equiv_F g \quad \text{iff} \quad \{x \mid f(x) = g(x)\} \in F.$$

It is easy to see that this is an equivalence relation and we denote by f/F or by $[f]$ the equivalence class of f.

The *recursive ultrapower* R/F is the model whose domain is the set of equivalence classes with respect to F. N is embedded in R/F by the identification of n with the function which is constantly n. The operations $+$ and $\cdot$ are defined by:

$$[f] + [g] = [f + g], \qquad [f] \cdot [g] = [fg]$$

(where fg is pointwise multiplication).

It is easy to see that the definition is independent of the choice of f and g and it follows that R/F is a homomorphic image of R.

If F contains a singleton then $R/F = N$ and the homomorphism is an evaluation. Otherwise, we get a model which strictly includes N as the identity function I is not equivalent to any constant function.

2.2 We introduce order on R/F by

$$[f] < [g] \quad \text{iff} \quad \{x \mid f(x) < g(x)\} \in F$$

this definition is independent of the choice of f and g and the properties of this order are summarized by the following lemma whose proof is straightforward.

LEMMA.

a. $<$ *is a linear order on* R/F *and* N *is an initial segment.*

b. $[f] < [g]$ *is equivalent to* $\exists x([f] + x + 1 = [g])$.

Thus, the notion of a bounded formula has a natural interpretation in recursive ultrapowers. We shall see that these models are models of T while other homomorphic images of R fail to be so for trivial reasons.

2.3 The main property of recursive ultrapowers is:

THEOREM. *If* $\phi(\bar{x})$ *is a* Σ_1 *formula then for any n recursive function* $f_1 \cdots f_n$, $R/F \models \phi([f_1], \cdots [f_n])$, *iff the set* $\{x \mid N \models \phi(f_1(x), \cdots f_n(x))\}$ *includes a set in F. If, in addition,* ϕ *is recursive then this set itself must be in F.*

PROOF. (Note that we treat the recursive functions as having names in the language. It is not difficult to restate the theorem in the terminology introduced in the first section.)

First, we prove the theorem for bounded formulas. It holds by definition for atomic formulas and it follows immediately for Boolean combinations. We proceed by induction on the number of quantifiers.

Assume that $\phi = \exists z < [f_1] \psi([f_1], \cdots [f_n], z)$ and $R/F \models \phi$. Then for some $g \in R$ $R/F \models \psi([f_1], \cdots [f_n], [g]) \wedge ([g] < [f_1])$. By the induction assumption

$$\{x \mid \psi(f_1(x), \cdots f_n(x), g(x)) \wedge (g(x) < f_1(x))\} \in F.$$

But this set is included in the set:

$$\{x \,|\, \exists z [\psi(f_1(x), \cdots f_n(x), z) \wedge (z < f_1(x))]\} = A.$$

Assume on the other hand that $A \in F$. As ψ is recursive the following is a recursive function:

$$g(x) = \mu z(z < f_1(x) \wedge \psi(f_1(x), \cdots f_n(x), z))$$

(with the usual assumption that $z = f_1(x)$ if no such z exists). Then

$$\{x \mid \psi(f_1(x), \cdots f_n(x), g(x)) \wedge (g(x) < f_1(x))\} \in F$$

and again by the induction assumption $R/F \models \psi([f_1], \cdots [f_n], [g]) \wedge ([g] < [f_1])$. If the first quantifier is universal, i.e. $\phi = \forall z < [f_1]\psi([f_1], \cdots, [f_n], z)$, then $\neg\phi$ has the same number of quantifiers and the first one existential. Therefore, the proof above holds for $\neg\phi$. Let B be the set $\{x \mid \forall z[z < f_1(x) \rightarrow \psi(f_1(x), \cdots, f_n(x), z)]\}$. B is recursive so that $B \in F$ or $\bar{B} \in F$. The claim now follows as $\bar{B} \in F$ iff $R/F \models \neg\phi$. (Note that in the proof above, we assumed that in $\exists x < t\psi$ t was a variable and not a general term. This is enough as $\exists x < t\psi$ can be replaced by $\exists x < y(\psi \wedge y = t)$).

To conclude the proof, we assume that $\phi(\bar{x}) = \exists y_1 \cdots \exists y_k \psi(\bar{x}), y_1 \cdots y_k)$ where ψ is bounded. If $R/F \models \phi([f_1], \cdots, [f_n])$ then for some $g_1 \cdots g_k$, $R/F \models \psi([f_1], \cdots, [f_n], [g_1], \cdots, [g_k])$. By the first part of the proof,

$$A = \{x \mid \psi(f_1(x), \cdots, f_n(x), g_1(x), \cdots, g_k(x))\} \in F.$$

Hence, $\{x \mid \phi(f_1(x), \cdots, f_n(x))\} \supset A$.

On the other hand, if $A \in F$ and for every $x \in A$ $N \models \exists y_1 \cdots y_k \psi(f_1(x), \cdots, f_n(x), y_1, \cdots, y_k)$, then for every $x \in A$

$$N \models \exists z \exists y_1 < z \cdots \exists y_k < z\psi(f_1(x), \cdots, f_n(x), y_1 \cdots y_k),$$

we define

$$g(x) = \begin{cases} \mu z(\exists y_1 < z \cdots \exists y_k < z\psi(f_1(x), \cdots f_n(x), y_1 \cdots y_k) & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

$g(x)$ is recursive and

$$A \subset \{x \mid \exists y_1 < g(x) \cdots \exists y_k < g(x)\psi(f_1(x), \cdots f_n(x), y_1 \cdots y_k)\}.$$

This formula is bounded so that

$$R/F \models \exists y_1 < [g] \cdots \exists y_k < [g] \psi([f_1], \cdots, [f_n], y_1 \cdots y_k).$$

q.e.d.

2.4 Corollary. *If F is a recursive ultrafilter then $R/F \models T$.*

Proof. Assume that $\phi \in T$ where $\phi \equiv \forall \bar{x} \exists y_1 \cdots y_k \psi(\bar{x}, y_1 \cdots y_k)$ and ψ is bounded. For every $f_1 \cdots f_n$,

$$N = \{x \mid \exists y_1 \cdots y_k \psi(f_1(x) \cdots f_n(x), y_1 \cdots y_k)\}.$$

By Theorem 2.3, $R/F \models \exists y_1 \cdots y_k \psi([f_1], \cdots, [f_n], y_1 \cdots y_k)$. As $f_1 \cdots f_n$ are arbitrary — $R/F \models \phi$.

q.e.d.

2.5 The following observation whose proof is immediate by 2.3 will be useful:

Lemma. *If $\phi(x,)$ is a Σ_1 formula which is the graph of the recursive function $f(x) = y$ then:*

$$R/F \models \phi([I], [f])$$

where I is the identity function. More generally — if $\psi(\bar{x}, y)$ is the graph of $h(\bar{x}) = y$ then for every $f_1 \cdots f_n$

$$R/F \models \psi([f_1], \cdots, [f_n], [h(f_1, \cdots, f_n)]).$$

2.6 Theorem. *Let $V(i, x, y)$ be the graph of a universal partial recursive function* (1.4). *The following formulas define $M - N$ in every recursive ultrapower M.*

$$\psi(z) \equiv \forall x \exists y [y < z \wedge V(y, [I], x)]$$

$$\bar{\psi}(z) \equiv \exists s \forall x \exists y [y < z \wedge V(y, s, x)]$$

Proof. If z is finite, then the partial recursive function described by V obtains only a finite number of values for a fixed s and for $0, 1, \cdots z - 1$. If on the other hand, z is infinite then for every recursive function f, there is an $i \in N$ (so that $i < z$) such that $V(i, x, y)$ is the graph of $f(x) = y$. By Lemma 2.5, $M \models V(i, [I], [f])$ and, therefore, $\psi(z)$ and $\bar{\psi}(z)$ hold.

q.e.d.

2.7 We denote by $N(x)$ the negation of $\bar{\psi}(x)$. $N(x)$ is a formula which defines N in every recursive ultrapower. Let K be the set of statements which hold in all recursive ultrapowers. Then for every statement ϕ,

$$N \models \phi \quad \text{iff} \quad K \models \phi^N$$

(where ϕ^N represents the relativization of ϕ to $N(x)$).
Thus, no consistent extension of K is an arithmetical set. N itself can be obtained as a trivial recursive ultrapower so that we have $K \subset (N)$, and indeed, $T(N) \equiv K \cup \{\forall x N(x)\}$. It follows also that K is a much richer theory than T as the last one is arithmethical. Finally, as we shall note in Section 4, there are many recursive ultrapowers which have arithmetical diagrams.

2.8 To conclude this section we discuss briefly other homomorphic images of R. Let σ be such a homomorphism onto a semiring S. Disregarding the trivial case, we can assume that $\sigma(1) = 1 -$ the unit of S. We show that if S has no zero divisors then σ determines a recursive ultrafilter F such that σ can be lifted to R/F.

For every recursive set A and its characteristic function χ_A we have:

$$\chi_A + \chi_{\bar{A}} = 1 \qquad \chi_A \cdot \chi_{\bar{A}} = 0,$$

and

$$\sigma(\chi_A) \mid \sigma(\chi_{\bar{A}}) = 1 \qquad \sigma(\chi_A) \cdot \sigma(\chi_{\bar{A}}) = 0.$$

We conclude that σ takes one of the functions to 0 and the other to 1. Hence, the following is an ultrafilter:

$$F = \{A \mid \sigma(\chi_A) = 1\}.$$

It remains now only to show that the definition, $\pi([f]) = \sigma(f)$, is independent of the choice of f. So assume that $[f] = [g]$ and $A = \{x \mid f(x) = g(x)\} \in F$. Hence, $\sigma(\chi_A) = 1$ and $\chi_A \cdot f = \chi_A \cdot g$. This yields:

$$\sigma(f) = \sigma(x_A)\sigma(f) = \sigma(\chi_A f) = \sigma(\chi_A g) = \sigma(g),$$

and our claim is proved.

Thus, any other homomorphic image with no zero divisor must identify two elements of some recursive ultrapower. This means that no order similar to that of N can be introduced because it is always the case that $\exists x \exists y (x = x + y + 1)$.

3. The models of T

In this section, the relations between recursive ultrapowers and general models of T are investigated.

3.1 Let M be a model of T, and $c \in M$. From Sec. 1, we know that the minimal model which contains c in M is the closure of c under recursive functions. This closure is absolute—it is the same in every extension of M which is a model of T and in any such submodel that contains c. We shall see that this closure is a recursive ultrapower so that such ultrapowers are in a sense the elementary compounds of which models of T (and $T(N)$) are constructed.

For M and c, as above, let F be the filter:

$$F = \{\phi(x) \mid M \models \phi(c),\ \phi \text{ is recursive}\}$$

(we identify recursive predicates with recursive sets).

THEOREM. *The correspondence $f/F \to f(c)$ is an isomorphism between R/F and the minimal models of T generated by c.*

PROOF. The correspondence is well defined: if $[f] = [g]$, then the predicate $f(x) = g(x)$ (represented by a recursive formula) is in F so that $f(c) = g(c)$. Similarly, one shows that the correspondence is one to one.

From the definition, it is clear that the image of R/F is exactly the closure of c under recursive functions.

The operations are preserved: If (for example), $[f] + [g] = [h]$, then the predicate $f(x) + g(x) = h(x)$ is in F so that $f(c) + g(c) = h(c)$.

q.e.d.

3.2 On the other hand, we shall see that the recursive ultrapowers are rich enough in structure so that every countable model of T can be embedded in such a model. Furthermore, we shall characterize the class of countable models of T by means of recursive ultrapowers.

THEOREM. *If M is a countable model of T then there is a recursive ultrapower that includes M and in which M is existentially complete.*

PROOF. Let $a_1, \cdots a_n, \cdots n < \omega$ be any ordering of M and let $\phi(i, x, y)$ be a recursive formula that describes the function $(x)_i = y$ (y is the highest power to which the i'th prime number divides x). Let $T(M)$ be the complete theory of M with the names of the elements of M. Finally let c be a new constant and denote by $\bar{T}$ the theory

$$\bar{T} = T(M) \cup \{\phi(i, c, a_i) \mid i \in N\}.$$

$\bar{T}$ is consistent as M can be turned into a model of any finite part of it by the appropriate choice of c. Hence $\bar{T}$ has a model M_1. Let M_0 be the closure of c under recursive functions in M_1, so that M_0 is a recursive ultrapower. As all models above are models of T, ϕ defines the same function in all of them, so that $M \subset M_0$. As M_1 is an elementary extension of M, M is existentially complete in every submodel of M_1 and in particular in M_0.

q.e.d.

The converse of the theorem is easy:

THEOREM. *If $M_0 \models T$ and if M is existentially complete in M_0, then $M \models T$.*

PROOF. This follows from the fact that T is inductive. Let ϕ be in T. By 1.7.1, we can assume that $\phi \equiv \forall \bar{x} \exists y_1 \cdots y_k \psi(\bar{x}, y_1, \cdots, y_k)$ where ψ is quantifier free. For arbitrary $\bar{a} \in M$, $M_0 \models \exists y_1 \cdots y_k \psi(\bar{a}, y_1 \cdots y_k)$ and as M is existentially complete the same holds in M. Thus $M \models \phi$.

q.e.d.

3.3 COROLLARIES

a) *The countable models of T are exactly the existentially complete submodels of the recursive ultrapowers, in particular:*

b) *All countable models of full Arithmetic are submodels of homomorphic images of the semiring of recursive functions.*

Corollary b) is the main result of [3] where a more complicated proof replaced the use of Matijasevic's theorem.

3.4 We present another characterization of the models of T in terms of recursive ultrapowers. Here we use an idea of H. J. Keisler [4].

DEFINITIONS

a) Let G be a collection of partitions of N. G is a *partition filter* if the common refinement of two partitions in G is again in G, and if any partition which is coarser than a partition in G is also in G.

b) For every function $f:N \to N$ we denote by $\langle f \rangle$ the partition $x \equiv y$ iff $f(x) = f(y)$.

c) If F is a recursive ultrafilter and G a partition filter then the *limit ultrapower* $R/F/G$ is the submodel of R/F of (the classes of) the functions f for which $\langle f \rangle$ agrees with some partition of G on some set in F.

3.5 THEOREM. *If f is a recursive ultrapower and G a partition filter then $M = R/F/G$ is existentially complete in R/F.*

PROOF. Assume that $R/F \models \phi([f_1], \cdots [f_n], [g_1], \cdots [g_k])$ where ϕ is quantifier free and $[f_i] \in M$ $i = 1 \cdots m$. Then:

$$\{x \mid N \models \phi(f_1(x), \cdots f_n(x),\ g_1(x), \cdots g_k(x))\} = A \in F,$$

we define $h(x)$:

$$h(x) = \begin{cases} \mu y \phi(f_1(x) \cdots f_n(x),\ (y)_1, \cdots (y)_k) & \text{if } x \in A \\ 1 & x \notin A. \end{cases}$$

h is recursive and so are the functions $g'_i(x) = (h(x))_i$ $i = 1 \cdots k$. For every $x \in A$

$$N \models \phi(f_1(x), \cdots f_n(x), g'_1(x) \cdots g'_k(x))$$

so that $R/F \models \phi(([f_1], \cdots [f_n), [g'_1] \cdots [g'_k])$.
By the definition of h if each one of the functions $f_1 \cdots f_n$ obtains the same value for the elements x and y in A then so do also $g'_1 \cdots g'_k$. Thus g'_i is coarser (on A) than the common refinement of $\langle f_1 \rangle \cdots \langle f_n \rangle$ so that $[g_i] \in M$ for $i = 1 \cdots k$.

q.e.d.

3.6 We have also the converse:

THEOREM. *Let R/F be a recursive ultrapower and M a submodel which is existentially complete in R/F. Let G be the partition filter generated by $\{\langle f \rangle \mid [f] \in M\}$. Then $R/F/G = M$.*

PROOF. Clearly $M \subset R/F/G$. So assume that $\langle g \rangle$ is in the filter, we shall prove that $[g] \in M$. By the definition of the filter, there are functions $f_1 \cdots f_n$

such that $[f_i] \in M$ $i = 1 \cdots n$ and $\langle g \rangle$ is coarser than the common refinement of $\langle f_1 \rangle \cdots \langle f_n \rangle$. Let $\phi_i(x, y)$ $i = 1 \cdots n$ and $\psi(x, y)$ be the graphs of $f_i(x)$ and $g(x)$ respectively described by existential formulas (which is possible by Matijasevic's theorem). By 2.5,

$$R/F \models \phi_1([I], [f_1]) \wedge \cdots \wedge \phi_n([I], [f_n]) \wedge \psi([I], (g]).$$

As M is existentially complete there are I' and g' such that $[I'] \in M$, $[g'] \in M$ and

$$R/F \models \phi_1([I'], [f_1]) \wedge \cdots \wedge \phi_n([I'], [f_n]) \wedge \psi([I'], [g']).$$

Thus the following set is in F:

$$\{x \mid \phi_1(I'(x), f_1(x)) \wedge \cdots \wedge \phi_n(I'(x), f_n(x)) \wedge \psi(I'(x), g'(x))\}$$
$$= \{x \mid f_1(I'(x)) = f_1(x) \wedge \cdots \wedge f_n(I'(x)) = f_n(x) \wedge g(I'(x)) = g(x)\}.$$

We complete the proof by showing that on this set, g and g' agree so that $[g] = [g']$. Indeed if x is in this set then by the first n conjuncts x and $I'(x)$ are in the same classes of $\langle f_i \rangle$ $i = 1 \cdots n$. By assumption, they are also in the same class of $\langle g \rangle$ so that $g(I'(x)) = g(x)$. By the last conjunct, this equals to $g'(x)$ so that $g(x) = g'(x)$.

q.e.d.

3.7 Thus the limit recursive ultrapowers are exactly the existentially complete submodels of recursive ultrapowers. By 3.3, we have:

COROLLARY. The countable models of T are exactly the limit recursive ultrapowers.

4. Examples of recursive ultrapowers

In this section, three particular kinds of recursive ultrapowers are briefly discussed.

4.1 *Comaximal ultrapowers.* Let S be a maximal (r-maximal) set, i.e., S is r.e, its complement $\bar{S}$ is infinite and for every r.e (recursive) set A either $A \cap \bar{S}$ is finite or $\bar{S} - A$ is finite Let F be the collection:
$F = \{A \mid A$ is recursive and $\bar{S} - A$ is finite$\}$.

It is easy to see that F is a recursive ultrafilter, $f \equiv_F g$, iff $f(x) = g(x)$ a.e on $\bar{S}$ (i.e. for all but a finite number of elements of $\bar{S}$, $f(x) = g(x)$) and $[f] + [g] = [h]$ iff $f(x) + g(x) = h(x)$ a.e on $\bar{S}$. Thus, R/F is the model $R/\bar{S}$ which was suggested by S. Tennenbaum [1] and investigated by M. Lerman in [5] and [6].

4.2 Remark 2.7 disproves the conjecture by A. Nerode in [5]; the theory of a comaximal ultrapower is never an arithmetical set. On the other hand, these models provide an example of a recursive ultrapower with an arithmetical diagram. It is easy to define by a formula the equivalence relation modulo $\bar{S}$ on the indices of (total) recursive functions, to choose by a formula a set of representatives and to define the operations on this set by a formula.

4.3 M. Lerman showed in [6] that two comaximal ultrapowers which are elementary equivalent are isomorphic. We can strengthen this result:

If S is r-maximal and F a recursive ultrafilter then $R/\bar{S} \equiv R/F$ iff they are isomorphic.

We outline the proof that uses the predicate $N(x)$ which defines N in both models. For every recursive formula $\phi(x)$ $R/\bar{S} \models \phi([I])$ iff:

$$R/\bar{S} \models \exists j [N(j) \wedge \forall i (N(i) \wedge j < i \wedge i \in \bar{S} \rightarrow \phi(i))].$$

Using the universal r.e. predicate $F(i, x)$ we can describe the recursive type of the identity $[I]$ by a single formula. Furthermore, the fact that an element generates the whole model is expressible by a formula. Thus, R/F is also generated by an element with the same recursive type and it is easy to construct an isomorphism that takes $[I]$ to this element.

4.4 *Minimal models.* Ordering the recursive functions $f_1 \cdots f_n \cdots$, it is easy to get a decreasing sequence of infinite recursive sets $A_1 \cdots A_n \cdots$ such that each one of the functions $f_1 \cdots f_n$ is either constant or monotone on A_n. Extending the sequence to a recursive ultrafilter we get a model of T whose only submodel which is also a model of T is N.

A simple splitting argument at the construction of the sequence shows that there are $2^{\aleph_0}$ such models which are pairwise non-isomorphic.

5.5 *Existentially complete ultrapowers.* Every infinite r.e set includes an infinite recursive subset. It is easy to construct a recursive ultrafilter F such that every r.e set includes some member of F or is disjoint to some member of F.

CLAIM. *If F is such a filter then R/F is existentially complete.*

We outline the proof: Assume that $R/F \models \neg\phi([f_1], \cdots [f_n])$ where ϕ is existential. By 2.5, $\{x \mid \phi(f_1(x) \cdots (x))\}$ does not include a member of F. By the construction of F there is some $A \in F$ such that A is disjoint to this set. A is described by a recursive formula $\psi(x)$ so that

$$N \models \psi(x) \rightarrow \neg\phi(f_1(x), \cdots f_n(x)).$$

Clearly the same holds also in T (where the functions are replaced by formulas that describe them). If M is a model of T that includes R/F then $M \models \psi([I])$ and $M \models f_i([I]) = [f_i]$; hence, $M \models \neg\phi([f_1] \cdots [f_n])$. This proves the claim.

4.6 In models as above there is a simpler definition of N. Let S be a simple set—i.e. an r.e set whose complement is an infinite set which does not include an infinite r.e subset. Let $s(x)$ be a formula that describes S and assume that $R/F \models \neg s\ ([f])$. Then $\{x \mid f(x) \in S\} \not\in F$ and by the construction of F we get a set $A \in F$ which is disjoint from this set. Hence, $f(A) \subset \bar{S}$ and $F(A)$ is finite. Thus f is constant on some set of F, and $[f] \in N$. We conclude that in R/F every non-standard element satisfies the formula $a(x)$ and the following formula describes $R/F - N$:

$$\psi(z) \equiv \forall y(y > z \rightarrow s(y)).$$

5. Arithmetical ultrapowers

A similar theory can be developed for models of full Arithmetic in terms of elementary substructures rather than of plain submodels. Indeed, much of it was done by A. Robinson in [8]. We outline the construction:

5.1 Arithmetical ultrapowers are defined similarly to recursive ultrapowers using maximal filters of arithmetical sets and the family of arithmetically definable functions.

5.2 Let M be a model of full Arithmetic and $A \subset M$. The closure of A under arithmetical functions yields a minimal elementary submodel of M which includes A, (if A is a singleton we get an arithmetical ultrapower). From this it follows also that the intersection of any number of elementary submodels of a model of Arithmetic is again an elementary submodel.

5.3 Every countable model of full arithmetic can be elementary embedded in an arithmetical ultrapower.

5.4 A limit arithmetical ultrapower is an elementary submodel of this ultrapower. Conversely, every elementary submodel of an arithmetical ultrapower determines a partition filter so that the corresponding limit ultrapower is this submodel.

5.5 Hence the countable models of full Arithmetic are just the limit arithmetical ultrapowers.

5.1–5.2 were proved by A. Robinson [8] and 5.3–5.5 are proved similarly to 3.2, 3.5, 3.6 and 3.7.

Added in proof. A more detailed discussion of 1.1–1.6 which were remarked about here without proofs can be found in *Forcing, Arithmetic and Division Rings* by W. H. Wheeler and the present author which will be published by Springer Verlag. Examples 4.4 and 4.5 are also taken from there.

References

1. S. Feferman, D. Scott and S. Tennenbaum. *Models of arithmetic through function rings*, Notices of the Amer. Math. Soc. **173** (1959).

2. H. Gaifman, *A note on models and submodels of arithmetic*, Conference in Mathematical Logic, London, 1970, Springer Verlag 1972, pp. 128–144.

3. J. Hirschfeld, *Models of Arithmetic and the semiring of recursive functions*, Victoria Symposium on Non-Standard Analysis, Springer Verlag, 1974, pp. 99–105.

4. H. J. Keisler, *Limit ultrapowers*, Trans. Amer. Math. Soc. **107** (1972), 382–408.

5. M. Lerman, Doctoral Dissertation, Cornell University, (1968).

6. M. Lerman, *Recursive functions modulo co-r-maximal sets*, Trans. Amer. Math. Soc. **148** (1970), 429–444.

7. Yu. V. Matijasevic, *Diophantine representation of recursively enumerable predicates* Proceeding of the second Scandinavian logic symposium, North-Holland (1971).

8. A. Robinson, *Model theory and non-standard arithmetic*, Infinitistic methods, Proceeding of the symposium in Warsaw (1959), 265–302.

9. J. R. Shonfield, *Mathematical logic*, Addison-Weseley, 1967.

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